A father protocol for quantum broadcast channels

Frédéric Dupuis and Patrick Hayden

***Abstract*— We present a new protocol for quantum broadcast channels based on the fully quantum Slepian-Wolf protocol. The protocol yields an achievable rate region for entanglement-assisted transmission of quantum information through a quantum broadcast channel that can be considered the quantum analogue of Marton's region for classical broadcast channels. The protocol can be adapted to yield achievable rate regions for unassisted quantum communication and for entanglement-assisted classical communication. Regularized versions of all three rate regions are provably optimal.**

***Index Terms*— quantum information, broadcast channels**

I. Introduction

DISCRETE memoryless broadcast channels are channels with one sender and multiple receivers modelled as a probability transition matrix $p(y_1, \ldots, y_n|x)$. There are many interesting tasks that one may want to perform using these channels, such as sending common messages to all the users, sending separate information to each user, sending data to each user privately, or some combination of these tasks. Here we shall focus only on sending separate data, and most of our discussions will only involve channels with two receivers.

These channels were first introduced by Tom Cover in [1], where he suggested that it may be possible to use them more efficiently than by timesharing between the different users. Since then, several results concerning broadcast channels have been found, such as the capacity of degraded broadcast channels (see, for example, [2]).

One particularly relevant result on classical broadcast channels is due to Marton [3]: given a probability distribution $p(x, u_1, u_2) = p(u_1, u_2)p(x|u_1, u_2)$, the following rate region is achievable for the general two-user broadcast channel $p(y_1, y_2|x)$:

$$\begin{aligned} R_1 &\leqslant I(U_1; Y_1) \\ R_2 &\leqslant I(U_2; Y_2) \\ R_1 + R_2 &\leqslant I(U_1; Y_1) + I(U_2; Y_2) - I(U_1; U_2) \end{aligned} \tag{1}$$

It is conjectured that this characterizes the capacity region of general broadcast channels, but despite considerable efforts, no one has been able to prove a converse theorem.

The quantum generalization of broadcast channels was first studied in [4] as part of a recent effort to develop a network quantum information theory [5], [6], [7], [8], [9], [10], [11], [12]. In [4], the authors derived three classes of results, the first one about channels with a classical input and quantum outputs, the second one about sending a common classical message while sending quantum information to one receiver, and the third about sending qubits to one receiver while establishing a GHZ state with the two receivers.

In this paper, we study quantum broadcast channels using a different approach. Over the past few years, several results in quantum Shannon theory have been unified and simplified by the introduction of the mother and father protocols [13] and, more recently, by the fully quantum Slepian-Wolf (FQSW) protocol [14] [15]. Thus, a whole panoply of protocols, such as the quantum reverse Shannon theorem [16], the Lloyd-Shor-Devetak (LSD) theorem [17] [18] [19], one-way entanglement distillation [20], and distributed compression [14], can be derived from the FQSW protocol in various ways. The results presented here are of the same flavour: we will derive a new coding theorem for general quantum broadcast channels using the FQSW theorem. The new protocol corresponds to a father protocol for broadcast channels: the sender transmits independent quantum information to each of the receivers using entanglement he already shares with each of them. Like the original father protocol, it easily can be transformed into a protocol for entanglement-assisted transmission of classical information via superdense coding or into a protocol for unassisted transmission of qubits by using part of the transmission capacity to send the needed entanglement.

The paper is structured as follows. After introducing our notation and giving some background on quantum information in section II, as well as a quick review of the FQSW protocol in section III, we present a high-level overview of the protocol in section IV. We then state and prove a one-shot version of the protocol in section V, and then move on to the i.i.d. version of the protocol in section VI. Finally, we conclude in section VII.

II. Background and notation

Quantum subsystems will be labelled by capital letters A, B, etc; and their associated Hilbert spaces will be denoted by $\mathcal{H}_A$, $\mathcal{H}_B$, etc. When necessary, we will use superscripts to indicate which subsystems a pure or mixed state is defined on; for instance, $|\psi\rangle^{AB} \in \mathcal{H}_{AB}$. We will abbreviate $\dim \mathcal{H}_A$ by $|A|$.

Quantum operations will also be written using superscripts to denote the input and output systems; for example, $U^{A'\to B}$ is an operator which takes the quantum subsystem A' as input and sends its output onto subsystem B. Generally, isometries will be written as U, V, and so forth, whereas quantum channels (also known as superoperators, or completely positive trace-preserving maps) will be written using calligraphic letters, such as $\mathcal{N}^{A'\to B}$. A quantum broadcast channel is a quantum channel with one input subsystem and two or more output subsystems.

Note that a quantum channel can always be extended into an isometry by adding another output subsystem which represents

F. Dupuis is with the Université de Montréal. email: dupuisf@iro.umontreal.ca
P. Hayden is with McGill University, email: patrick@cs.mcgill.ca

the environment of the channel. This isometric extension implements exactly the same operation as the original channel if we trace out the environment subsystem. The isometric extension of $\mathcal{N}^{A'\to B}$ will be denoted by $U_{\mathcal{N}}^{A'\to BE}$, where E is the environment.

We also use the symbol $\cdot$ in the form $A \cdot B := ABA^\dagger$ to denote conjugation of B by A. This will allow us to avoid writing symbols twice when applying several operators to a quantum state.

We will also denote a "standard" entangled pair between subsystems S and S' of equal size as $|\Phi\rangle^{SS'} = \sum_{i=0}^{|S|} |ii\rangle^{SS'}$, where the $|i\rangle^S$ and $|i\rangle^{S'}$ are some standard basis on S and S'.

We will often use the *trace norm* of a hermitian matrix M, defined to be $\|M\| := \operatorname{Tr}|M|$. It is particularly useful because it induces a statistically important metric on the space of quantum states; we call the quantity $\|\rho - \sigma\|$ the *trace distance* between ρ and σ .

The von Neumann entropy of a density operator ρ^A will be denoted $H(\rho^A) = H(A)_\rho$. The quantum mutual information of ρ^{AB} is the function $I(A;B)_\rho = H(A)_\rho + H(B)_\rho - H(AB)_\rho$ while the coherent information is the function $I(A\rangle B)_\rho = H(B)_\rho - H(AB)_\rho$.

Finally, we will say that two families of states ψ and φ parametrized by their size n are asymptotically equal (denoted $\psi \approx_{(a)} \varphi$) if $\|\psi - \varphi\|$ vanishes as $n \to 0$. See Appendix I for a formal definition.

A. Achievable rates and the capacity region

Here we define what we mean by *achievable rates* and the *capacity region* of a quantum broadcast channel $\mathcal{N}^{A'\to B_1B_2}$ for entanglement-assisted transmission. We define a $(Q_1, Q_2, n, \varepsilon)$-code to consist of an encoding isometry $W^{A_1\tilde{A}_1A_2\tilde{A}_2\to\hat{A}A'^{\otimes n}}$ and two decoding isometries $V_1^{B_1^{\otimes n}\tilde{B}_1\to\bar{B}_1\hat{B}_1}$ and $V_2^{B_2^{\otimes n}\tilde{B}_2\to\bar{B}_2\hat{B}_2}$ such that

$$\left\| \left((V_2 V_1 U_{\mathcal{N}}^{\otimes n} W) \cdot \varphi \right) - \hat{\psi}^{\hat{B}_1\hat{B}_2 E\hat{A}} \otimes \Phi^{R_1\bar{B}_1} \otimes \Phi^{R_2\bar{B}_2} \right\| \leqslant \varepsilon$$

where $|\varphi\rangle = |\Phi\rangle^{R_1A_1} \otimes |\Phi\rangle^{\tilde{A}_1\tilde{B}_1} \otimes |\Phi\rangle^{R_2A_2} \otimes |\Phi\rangle^{\tilde{A}_2\tilde{B}_1}$ and $\hat{\psi}^{\hat{B}_1\hat{B}_2E\hat{A}}$ is a pure state, and where $\log|A_1| = Q_1$ and $\log|A_2| = Q_2$. A_1 and A_2 represent the systems that Alice wants to send to Bob 1 and Bob 2 respectively, and $\tilde{A}_1\tilde{B}_1$ and $\tilde{A}_2\tilde{B}_2$ are the EPR pairs Alice shares with the two receivers. Note that in practice, the encoding and decoding operations can be any completely positive, trace-preserving maps. We choose to implement these maps using isometries because this will prove much more convenient below.

A rate point (Q_1, Q_2) is *achievable* if there exists a sequence of $(Q_1, Q_2, n, \varepsilon_n)$-codes such that $\varepsilon_n \to 0$ as $n \to \infty$. The *capacity region* of the channel $\mathcal{N}$ is the closure of the union of all achievable rate points.

The unassisted quantum capacity region for $\mathcal{N}$ is defined in the same way, except that the protocol begins without any entanglement between Alice and Bob 1 or Alice and Bob 2. Formally, the definitions are identical except that in the unassisted case, the systems $\tilde{A}_1$, $\tilde{B}_1$, $\tilde{A}_2$ and $\tilde{B}_2$ are 1-dimensional or, equivalently, non-existent.

III. The FQSW protocol

Before presenting our protocol, we first give a quick overview of the fully quantum Slepian-Wolf protocol. Suppose Alice and Bob hold a mixed state ρ^{AB}. We introduce a reference system R to purify the state; the resulting state is thus $|\psi\rangle^{ABR}$. Alice would like to transfer her state to Bob by sending him as few qubits as possible. The FQSW theorem states that Alice can do this by first applying a unitary transformation to her entire share of the state (a random unitary selected according to the Haar measure will do), splitting her share into two subsystems $\bar{A}$ and $\hat{A}$, and then sending $\hat{A}$ to Bob.

Note that this scheme works provided that the subsystems $\bar{A}$ and R are in a product state after applying the random unitary: since Bob holds the purifying system of $\bar{A}R$, there exists a local unitary that Bob can apply to turn his purifying system into separate purifying systems of the two subsystems. The purifying system of R is exactly the original state that Alice wanted to send to Bob, and $\bar{A}$ together with its purifying system is an EPR pair shared by Alice and Bob. This last feature is an added bonus of the protocol: Alice and Bob get some free entanglement at the end.

It is possible to calculate how close $\bar{A}$ and R are to being in a product state. Here, we are particularly interested in the special case where $\rho^A = \frac{\mathbb{I}}{|A|}$. The result of the calculation is the following (see [14] for details):

$$\int_{\mathrm{U}(A)} \left\| \rho^{\bar{A}R}(U) - \frac{\mathbb{I}^{\bar{A}}}{|\bar{A}|} \otimes \rho^R \right\|_1^2 dU \leqslant \frac{|A||R|}{|\hat{A}|^2} \operatorname{Tr}\left[\left(\psi^{AR}\right)^2 \right] \tag{2}$$

Since the inequality holds for the average over choices of U, there must exist at least one U that satisfies it.

Another special case of interest is when the initial state is an i.i.d. state of the form $(|\psi\rangle^{ABR})^{\otimes n}$. In this case, it can be shown that as long as $\log|\hat{A}| \geqslant n[\frac{1}{2}I(A;R) + \delta]$, then

$$\varphi^{\bar{A}R^{\otimes n}} \approx_{(a)} \frac{\mathbb{I}^{\bar{A}}}{|\bar{A}|} \otimes \varphi^{R^{\otimes n}} \tag{3}$$

where $\varphi^{\bar{A}\hat{A}B^{\otimes n}R^{\otimes n}}$ is the result of applying the random unitary to $(\psi^{ABR})^{\otimes n}$, and $\delta > 0$.

IV. Overview of the protocol

Let's suppose Alice would like to send the maximally mixed system A_1 (which is purified by R_1) to Bob 1, and A_2 to Bob 2 using n instances of the quantum broadcast channel $\mathcal{N}^{A'\to B_1B_2}$. In addition, she has shared EPR pairs with both of them, represented by systems $\tilde{A}_1\tilde{B}_1$ and $\tilde{A}_2\tilde{B}_2$. We represent the channel by its isometric extension $U_{\mathcal{N}}^{A'\to B_1B_2E}$. Alice encodes her information using the encoding isometry $W^{A_1\tilde{A}_1A_2\tilde{A}_2\to A'\hat{A}}$; A' is then transmitted through the channel, and $\hat{A}$ is discarded (discarding a subsystem will turn out to be useful when discussing the i.i.d. case). Thus, after using the channel, the state of the system is $|\psi\rangle = U_{\mathcal{N}}^{\otimes n} W|\varphi\rangle$, where $|\varphi\rangle = |\Phi\rangle^{R_1A_1} \otimes |\Phi\rangle^{\tilde{A}_1\tilde{B}_1} \otimes |\Phi\rangle^{R_2A_2} \otimes |\Phi\rangle^{\tilde{A}_2\tilde{B}_2}$. See Figure 1 for a diagram illustrating this.

In order for Bob 1 to be able to decode, we have to make sure that R_1 is in a product state with everything else that

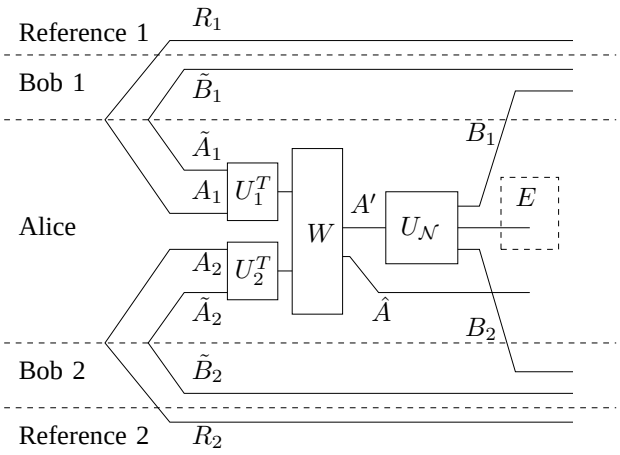

Fig. 1. Diagram illustrating the one-shot version of the protocol.

Bob 1 doesn't have access to, namely $R_2B_2\tilde{B}_2E\hat{A}$. Likewise, R_2 must be in a product state with $R_1B_1\tilde{B}_1E\hat{A}$. This is accomplished by applying an FQSW random unitary on $R_1\tilde{B}_1$ and another on $R_2\tilde{B}_2$, where R_1 and R_2 play the role of the system that stays behind. Now, since the FQSW unitary is applied to one end of a maximally entangled state, we can have the same effect by applying its transpose to the other end.

V. One-shot version

We first prove a generic "one-shot" version of our theorem which works for general states and channels; we will then use it to derive an achievable rate region for the case of many independent uses of the channel.

Theorem 1: For every encoding isometry $W^{A_1\tilde{A}_1A_2\tilde{A}_2\to A'\hat{A}}$, there exist isometries $U_1^{A_1\tilde{A}_1}$, $U_2^{A_2\tilde{A}_2}$, $V_1^{B_1\tilde{B}_1\to\bar{B}_1\hat{B}_1}$, and $V_2^{B_2\tilde{B}_2\to\bar{B}_2\hat{B}_2}$ such that

$$\begin{aligned}&\left\|\left((V_2V_1U_{\mathcal{N}}WU_2^TU_1^T)\cdot\varphi\right)-\hat{\psi}^{\hat{B}_1\hat{B}_2E\hat{A}}\otimes\Phi^{R_1\bar{B}_1}\otimes\Phi^{R_2\bar{B}_2}\right\|\\&\leqslant 4\left\{\frac{|R_1||\tilde{B}_1||R_2\tilde{B}_2B_2E\hat{A}|}{|\tilde{B}_1|^2}\operatorname{Tr}[(\psi^{R_1\tilde{B}_1R_2\tilde{B}_2B_2E\hat{A}})^2]\right\}^{\frac{1}{4}}\\&+2\left\{\frac{|R_2||\tilde{B}_2||R_1\tilde{B}_1B_1E\hat{A}|}{|\tilde{B}_2|^2}\operatorname{Tr}[(\psi^{R_2\tilde{B}_2R_1\tilde{B}_1B_1E\hat{A}})^2]\right\}^{\frac{1}{4}}\end{aligned}\tag{4}$$

where $|\varphi\rangle=|\Phi\rangle^{R_1A_1}\otimes|\Phi\rangle^{\tilde{A}_1\tilde{B}_1}\otimes|\Phi\rangle^{R_2A_2}\otimes|\Phi\rangle^{\tilde{A}_2\tilde{B}_1}$, and $\hat{\psi}^{\hat{B}_1\hat{B}_2E\hat{A}}$ is a pure state uniquely determined by the protocol.

Proof: Let ψ be defined as in section IV; applying formula (2) yields:

$$\begin{aligned}&\int_{\mathbb{U}(R_1\tilde{B}_1)}\left\|\sigma^{R_1R_2\tilde{B}_2B_2E\hat{A}}(U)-\frac{\mathbb{I}^{R_1}}{|R_1|}\otimes\sigma^{R_2\tilde{B}_2B_2E\hat{A}}\right\|_1^2dU\\&\leqslant\frac{|R_1||\tilde{B}_1||R_2\tilde{B}_2B_2E\hat{A}|}{|\tilde{B}_1|^2}\operatorname{Tr}\left[(\psi^{R_1\tilde{B}_1R_2\tilde{B}_2B_2E\hat{A}})^2\right].\end{aligned}\tag{5}$$

and

$$\begin{aligned}&\int_{\mathbb{U}(R_2\tilde{B}_2)}\left\|\sigma^{R_2R_1\tilde{B}_1B_1E\hat{A}}(U)-\frac{\mathbb{I}_{R_2}}{|R_2|}\otimes\sigma^{R_1\tilde{B}_1B_1E\hat{A}}\right\|_1^2dU\\&\leqslant\frac{|R_2||\tilde{B}_2||R_1\tilde{B}_1B_1E\hat{A}|}{|\tilde{B}_2|^2}\operatorname{Tr}[(\psi^{R_2\tilde{B}_2R_1\tilde{B}_1B_1E\hat{A}})^2].\end{aligned}\tag{6}$$

This means that there exist unitaries $U_1^{R_1\tilde{B}_1}$ and $U_2^{R_2\tilde{B}_2}$ that satisfy the above inequalities. As mentioned before, since $R_1\tilde{B}_1$ and $R_2\tilde{B}_2$ are maximally entangled, we can achieve the same effect by applying U_1^T and U_2^T on $A_1\tilde{A}_1$ and $A_2\tilde{A}_2$ respectively.

Now, using Uhlmann's theorem [21] in the form of Lemma 2.2 of [22], we get that there exist decoding unitaries $V_1^{B_1\tilde{B}_1\to\bar{B}_1\hat{B}_1}$ and $V_2^{B_2\tilde{B}_2\to\bar{B}_2\hat{B}_2}$ such that

$$\begin{aligned}&\left\|\left((V_2V_1U_{\mathcal{N}}WU_2^TU_1^T)\cdot\varphi\right)-\hat{\psi}_1^{R_1\bar{B}_2\hat{B}_1\hat{B}_2E\hat{A}}\otimes\Phi^{R_1\bar{B}_1}\right\|_1\\&\leqslant 2\left\{\frac{|R_1||\tilde{B}_1||R_2\tilde{B}_2B_2E\hat{A}|}{|\tilde{B}_1|^2}\operatorname{Tr}[(\psi^{R_1\tilde{B}_1R_2\tilde{B}_2B_2E\hat{A}})^2]\right\}^{\frac{1}{4}}\end{aligned}\tag{7}$$

and

$$\begin{aligned}&\left\|\left((V_2V_1U_{\mathcal{N}}WU_2^TU_1^T)\cdot\varphi\right)-\hat{\psi}_2^{R_2\bar{B}_1\hat{B}_1\hat{B}_2E\hat{A}}\otimes\Phi^{R_2\bar{B}_2}\right\|_1\\&\leqslant 2\left\{\frac{|R_2||\tilde{B}_2||R_1\tilde{B}_1B_1E\hat{A}|}{|\tilde{B}_2|^2}\operatorname{Tr}[(\psi^{R_2\tilde{B}_2R_1\tilde{B}_1B_1E\hat{A}})^2]\right\}^{\frac{1}{4}}\end{aligned}\tag{8}$$

where $\hat{\psi}_1$ and $\hat{\psi}_2$ are some pure states.

To finish, we need the following lemma:

Lemma 1: If we have

$$\begin{aligned}\left\|\rho^{ABC}-\sigma^A\otimes\sigma^{BC}\right\|&\leqslant\varepsilon_1\\\left\|\rho^{ABC}-\tau^{AB}\otimes\tau^C\right\|&\leqslant\varepsilon_2\end{aligned}$$

then $\left\|\rho^{ABC}-\sigma^A\otimes\tau^B\otimes\tau^C\right\|\leqslant 2\varepsilon_1+\varepsilon_2$.

Proof:

$$\begin{aligned}&\left\|\rho^{ABC}-\sigma^A\otimes\tau^B\otimes\tau^C\right\|\\&\quad\leqslant\left\|\rho^{ABC}-\sigma^A\otimes\sigma^{BC}\right\|\\&\qquad+\left\|\sigma^A\otimes\sigma^{BC}-\sigma^A\otimes\tau^B\otimes\tau^C\right\|\\&\quad=\varepsilon_1+\left\|\sigma^{BC}-\tau^B\otimes\tau^C\right\|\\&\quad\leqslant\varepsilon_1+\left\|\sigma^{BC}-\rho^{BC}\right\|+\left\|\rho^{BC}-\tau^B\otimes\tau^C\right\|\\&\quad\leqslant 2\varepsilon_1+\varepsilon_2\end{aligned}$$

■

Applying this to our system, we get equation (4). ■

VI. I.I.D version

Theorem 2: Let $\mathcal{N}^{A'\to B_1B_2}$ be a quantum broadcast channel. Then the following rate region is achievable for

$|\psi\rangle^{A_1A_2B_1B_2DE} = U_{\mathcal{N}}^{A'\to B_1B_2E}|\phi\rangle^{A_1A_2A'D}$ where $|\phi\rangle$ is any pure state:

$$\begin{aligned} Q_1 &\leqslant \frac{1}{2} I(A_1;B_1)_\psi \\ Q_2 &\leqslant \frac{1}{2} I(A_2;B_2)_\psi \\ Q_1 + Q_2 &\leqslant \frac{1}{2}\left[I(A_1;B_1)_\psi + I(A_2;B_2)_\psi - I(A_1;A_2)_\psi\right]. \end{aligned} \tag{9}$$

Q_1 is the rate at which Alice sends qubits to Bob 1, and likewise for Q_2 for Bob 2. Note that including the D subsystem is equivalent to allowing $\phi^{A_1A_2A'}$ to be a mixed state; we find this formulation more convenient for our purposes.

Proof: To get this rate region, we must apply the one-shot theorem to an i.i.d. state. The main challenge is that for an arbitrary i.i.d. state of the form $|\phi^{\mathcal{N}}\rangle^{A_1^{\otimes n}A_2^{\otimes n}B_1^{\otimes n}B_2^{\otimes n}D^{\otimes n}E^{\otimes n}} = U_{\mathcal{N}}^{\otimes n}(|\phi\rangle^{A_1A_2A'D})^{\otimes n}$, the $A_1^{\otimes n}$ and $A_2^{\otimes n}$ subsystems can be correlated, and to apply the one-shot theorem, it is crucial that $A_1^{\otimes n}$ and $A_2^{\otimes n}$ be maximally mixed and decoupled in order to play the role of $R_1\tilde{B}_1$ and $R_2\tilde{B}_2$ respectively. (We use the term *decoupled* to indicate that the density operator of a composite quantum system is the product of the reduced density operators of its component systems. The analogous notion in probability theory is independence.)

We can remedy this situation by using the FQSW protocol to decouple $A_1^{\otimes n}$ and $A_2^{\otimes n}$. Whether we apply it to $A_1^{\otimes n}$ or to $A_2^{\otimes n}$, it will require us to remove $n[\frac{1}{2}I(A_1;A_2)+\delta]$ qubits, where $\delta > 0$ can be arbitrarily small. The removed qubits will play the role of $\hat{A}$ in the previous section. Suppose without loss of generality that we apply it to $A_1^{\otimes n}$ only. (This will correspond to one of the corner points of the region and therefore, by time-sharing, the entire region will be achievable.) Let the FQSW unitary be $W_1^{A_1^{\otimes n}\to\bar{A}_1\hat{A}_1}$ where $\bar{A}_1$ plays the role of the system that stays behind in FQSW, and $\hat{A}_1$ is the system that is discarded.

At the end of this process, it can be shown (see equation (3)) that the $\bar{A}_1$ subsystem of $W_1\cdot\phi^{\mathcal{N}}$ is asymptotically equal to the maximally mixed state. To get $A_2^{\otimes n}$ to also be maximally mixed, we can apply another FQSW unitary to it, and discard $n\delta$ qubits from it (where δ can be arbitrarily small); this also leaves $\bar{A}_2$ asymptotically equal to the maximally mixed state. Let this second FQSW unitary be $W_2^{A_2^{\otimes n}\to\bar{A}_2\hat{A}_2}$, and let $|\psi\rangle^{\bar{A}_1\bar{A}_2\hat{A}_1\hat{A}_2A'D^{\otimes n}}$ be

$$W_2^{A_2^{\otimes n}\to\bar{A}_2\hat{A}_2} W_1^{A_1^{\otimes n}\to\bar{A}_1\hat{A}_1}(|\phi\rangle^{A_1A_2A'D})^{\otimes n}.$$

Applying equation (3) to W_1 and W_2, we obtain that

$$\psi^{\bar{A}_1\bar{A}_2\hat{A}_2} \approx_{(a)} \frac{\mathbb{I}^{\bar{A}_1}}{|\bar{A}_1|}\otimes\psi^{\bar{A}_2\hat{A}_2} \tag{10}$$

$$\psi^{\bar{A}_2} \approx_{(a)} \frac{\mathbb{I}^{\bar{A}_2}}{|\bar{A}_2|} \tag{11}$$

Hence, we have that $\psi^{\bar{A}_1\bar{A}_2} \approx_{(a)} \frac{\mathbb{I}^{\bar{A}_1\bar{A}_2}}{|\bar{A}_1||\bar{A}_2|}$, confirming that $\bar{A}_1$ and $\bar{A}_2$ are indeed maximally mixed.

Now, let $|\varphi\rangle = |\Phi\rangle^{R_1A_1}\otimes|\Phi\rangle^{\bar{A}_1\tilde{B}_1}\otimes|\Phi\rangle^{R_2A_2}\otimes|\Phi\rangle^{\bar{A}_2\tilde{B}_1}$, where we identify $R_1\tilde{B}_1$ with $\bar{A}_1$ and $R_2\tilde{B}_2$ with $\bar{A}_2$. Since $\bar{A}_1\bar{A}_2$ is asymptotically equal to the maximally mixed state in both $|\psi\rangle$ and $|\varphi\rangle$, by Uhlmann's theorem there exists an isometry $W^{A_1\bar{A}_1A_2\bar{A}_2\to\hat{A}_1\hat{A}_2A'D^{\otimes n}}$ such that $|\psi_U\rangle = W|\varphi\rangle$ is asymptotically equal to $|\psi\rangle$. Note that we can use Theorem 1 directly on $|\varphi\rangle$ and the encoding unitary W. This means that there exist isometries $U_1^{A_1\bar{A}_1}$, $U_2^{A_2\bar{A}_2}$, $V_1^{B_1\tilde{B}_1\to\bar{B}_1\hat{B}_1}$, and $V_2^{B_2\tilde{B}_2\to\bar{B}_2\hat{B}_2}$ such that equation (4) is satisfied.

Now, define Π_F to be the projector onto the $\varepsilon(n)$-typical subspace of an arbitrary subsystem $F^{\otimes n}$, where $\varepsilon(n)$ can be chosen such that $\lim_{n\to\infty}\varepsilon(n) = 0$ (see Appendix II), and let $|\psi_T\rangle$ be the normalized version of

$$\begin{aligned} U_{\mathcal{N}}^{\otimes n\,\dagger}\Pi_{A_1A_2B_1DE}\Pi_{A_1A_2B_2DE} \\ \Pi_{A_2B_2DE}\Pi_{A_1B_1DE}\Pi_{A_2}\Pi_{A_1}U_{\mathcal{N}}^{\otimes n}|\psi\rangle. \end{aligned}$$

Since the only differences between $|\psi\rangle$ and $|\psi_T\rangle$ are the presence of different typical projectors, it is possible (see Appendix II) to choose $\varepsilon(n)$ such that $\lim_{n\to\infty}\varepsilon(n) = 0$ and such that the two states are asymptotically equal. (Note that the argument relies on the transitivity of asymptotic equality.) We will therefore select $\varepsilon(n)$ such that $\psi_U \approx_{(a)} \psi_T$.

We will now evaluate the right-hand side of (4) using $|\psi_T^{\mathcal{N}}\rangle = U_{\mathcal{N}}^{\otimes n}|\psi_T\rangle$ (where $\bar{A}_1$ will be split into R_1 and $\tilde{B}_1$ and likewise for $\bar{A}_2$). From basic properties of typical subspaces (see Appendix II), for sufficiently large n we effectively have:

$$|R_1||\tilde{B}_1| = |\bar{A}_1| \leqslant 2^{n[H(A_1)-\frac{1}{2}I(A_1;A_2)+\delta]} \tag{12}$$

and

$$|R_2\tilde{B}_2B_2DE\hat{A}| \leqslant 2^{n[H(A_2B_2DE)+\delta]}2^{n[\frac{1}{2}I(A_1;A_2)+\delta]} \tag{13}$$

as well as

$$\begin{aligned} &\mathrm{Tr}[((\psi_T^{\mathcal{N}})^{\bar{A}_1A_2^{\otimes n}B_2^{\otimes n}D^{\otimes n}E^{\otimes n}\hat{A}_1})^2] \\ &\qquad = \mathrm{Tr}[((\psi_T^{\mathcal{N}})^{A_1^{\otimes n}A_2^{\otimes n}B_2^{\otimes n}D^{\otimes n}E^{\otimes n}})^2] \\ &\qquad \leqslant \frac{2^{-n[H(A_1A_2B_2DE)-\delta]}}{1-\delta} \end{aligned} \tag{14}$$

and therefore, the first term of the RHS of (4) is

$$\begin{aligned} &4\left\{\frac{|R_1||\tilde{B}_1||R_2\tilde{B}_2B_2DE\hat{A}|}{|\tilde{B}_1|^2}\,\mathrm{Tr}\left[\left((\psi_T^{\mathcal{N}})^{R_1\tilde{B}_1R_2\tilde{B}_2B_2DE\hat{A}}\right)^2\right]\right\}^{\frac{1}{4}} \\ &\qquad\qquad \leqslant 2\left\{\frac{2^{n[I(A_1;A_2B_2DE)+3\delta]}}{(1-\delta)|\tilde{B}_1|^2}\right\}^{\frac{1}{4}} \end{aligned}$$

Assuming $|\tilde{B}_1| \geqslant 2^{n[I(A_1;A_2B_2DE)/2+2\delta]}$, we get

$$\begin{aligned} &4\left\{\frac{|R_1||\tilde{B}_1||R_2\tilde{B}_2B_2DE\hat{A}|}{|\tilde{B}_1|^2}\,\mathrm{Tr}\left[\left((\psi_T^{\mathcal{N}})^{R_1\tilde{B}_1R_2\tilde{B}_2B_2DE\hat{A}}\right)^2\right]\right\}^{\frac{1}{4}} \\ &\qquad\qquad \leqslant \frac{4\times 2^{-n\delta}}{(1-\delta)^{\frac{1}{4}}} \end{aligned}$$

Likewise, we can evaluate the second term of the right-hand side of equation (4) and obtain that we need $|\tilde{B}_2| \geqslant 2^{n[I(A_2;A_1B_1DE)/2+2\delta]}$ to make it vanish.

Now, since $|\psi_T^{\mathcal{N}}\rangle \approx_{(a)} U_{\mathcal{N}}^{\otimes n}|\psi_U\rangle$, if we had calculated the LHS of (4) using $U_{\mathcal{N}}^{\otimes n}|\psi_U\rangle$ instead of $|\psi_T^{\mathcal{N}}\rangle$, by the triangle

inequality, we could only have gotten a value that is at most larger by a vanishing term. Hence, we get that

$$(V_2 V_1 U_{\mathcal{N}}^{\otimes n} W U_2 U_1) \cdot \varphi \approx_{(a)} \hat{\psi}^{\hat{B}_1 \hat{B}_2 D E \hat{A}} \otimes \Phi^{R_1 \bar{B}_1} \otimes \Phi^{R_2 \bar{B}_2},$$

which means that the scheme works.

We can now easily verify that our conditions on $|\tilde{B}_1|$ and $|\tilde{B}_2|$ indeed correspond to the rates advertised in the statement of the theorem. First, we have

$$\begin{aligned} nQ_1 &= \log |R_1| \\ &= \log |\bar{A}_1| - \log |\tilde{B}_1| \\ &\leqslant n \left[H(A_1) - \frac{1}{2} I(A_1; A_2) - \frac{1}{2} I(A_1; A_2 B_2 D E) - \delta \right] \\ &= \frac{1}{2} n \left[I(A_1; B_1) - I(A_1; A_2) - \delta \right] \end{aligned}$$

and

$$\begin{aligned} nQ_2 = \log |R_2| &= \log |\bar{A}_2| - \log |\tilde{B}_2| \\ &\leqslant n \left[H(A_2) - \frac{1}{2} I(A_2; A_1 B_1 D E) - \delta \right] \\ &= \frac{1}{2} n \left[I(A_2; B_2) - \delta \right] \end{aligned}$$

where δ vanishes as $n \to \infty$. We can, of course, exchange the roles of Bob 1 and Bob 2; combining this with time-sharing gives the asymptotic rates given in (9). ∎

We can also calculate how much entanglement is needed between Alice and the two Bobs; let E_1 be the rate at which EPR pairs between Alice and Bob 1 are used during the protocol, and define E_2 similarly for Bob 2. We have

$$\begin{aligned} nE_1 &= \log |\tilde{B}_1| \\ &\geqslant n \left[\frac{1}{2} I(A_1; A_2 B_2 D E) + 2\delta \right] \\ nE_2 &= \log |\tilde{B}_2| \\ &\geqslant n \left[\frac{1}{2} I(A_2; A_1 B_1 D E) + 2\delta \right] \end{aligned} \tag{15}$$

A. Unassisted transmission

Note that a simple modification of this protocol allows us to achieve transmission of qubits without needing pre-shared entanglement. We can first let Alice establish initial entanglement with Bob 1 using the LSD theorem (ignoring Bob 2 during this phase of the protocol); likewise, she can establish initial entanglement with Bob 2. Then, they can use the entanglement-assisted protocol just shown for the rest of the transmission, using part of the rate to maintain their stock of entanglement, and using the surplus to transmit qubits. Since we only need to use a suboptimal protocol for the initial stage, the asymptotic rates will be unaffected. The asymptotic rates will be

$$\begin{aligned} \bar{Q}_1 &= Q_1 - E_1 \\ &\leqslant \frac{1}{2} I(A_1; B_1) - \frac{1}{2} I(A_1; A_2) - \frac{1}{2} I(A_1; A_2 B_2 D E) \\ &= I(A_1 \rangle B_1) - \frac{1}{2} I(A_1; A_2) \\ \bar{Q}_2 &= Q_2 - E_2 \\ &\leqslant \frac{1}{2} I(A_2; B_2) - \frac{1}{2} I(A_2; A_1 B_1 D E) \\ &= I(A_2 \rangle B_2) \end{aligned}$$

yielding, via time-sharing, the following rate region:

$$\begin{aligned} \bar{Q}_1 &\leqslant I(A_1 \rangle B_1) \\ \bar{Q}_2 &\leqslant I(A_2 \rangle B_2) \\ \bar{Q}_1 + \bar{Q}_2 &\leqslant I(A_1 \rangle B_1) + I(A_2 \rangle B_2) - \frac{1}{2} I(A_1; A_2) \end{aligned}$$

A detailed proof that this strategy works requires a slightly more careful analysis of the broadcast father protocol than we have done here. Specifically, it is straightforward to verify that the entanglement generated in the father can be produced such that it is within $O(2^{-n\alpha})$ in trace distance of the standard maximally entangled state, for some $\alpha > 0$. This ensures that the father protocol can be repeated a number of times polynomial in n, re-using some of the output entanglement at each step, without causing significant degradation in the quality of the entanglement.

B. Regularized converse

The rate region given in theorem 2 is indeed the capacity of quantum broadcast channels provided we regularize over many uses of the channel. It is important to remember, however, that regions defined by very different formulas can nonetheless agree after regularization, so the following theorem should be understood to be only a very weak characterization of the capacity.

Theorem 3: The entanglement-assisted capacity region of a quantum broadcast channel $\mathcal{N}^{A' \to B_1 B_2}$ is the convex hull of the union of all rate points (Q_1, Q_2) satisfying

$$\begin{aligned} Q_1 &\leqslant \frac{1}{2n} I(A_1; B_1^{\otimes n}) \\ Q_2 &\leqslant \frac{1}{2n} I(A_2; B_2^{\otimes n}) \\ Q_1 + Q_2 &\leqslant \frac{1}{2n} [I(A_1; B_1^{\otimes n}) + I(A_2; B_2^{\otimes n}) - I(A_1; A_2)] \end{aligned} \tag{16}$$

for some state of the form $|\psi\rangle^{A_1 A_2 B_1^{\otimes n} B_2^{\otimes n} D E^{\otimes n}} = U_{\mathcal{N}}^{\otimes n} |\phi\rangle^{A_1 A_2 A'^{\otimes n} D}$, where $|\phi\rangle$ is a pure state.

Proof: It is immediate from theorem 2 that the region is achievable. We now prove the converse.

Suppose that (Q_1, Q_2) is an achievable rate pair. That means that there exists a sequence of $(Q_1, Q_2, n, \varepsilon_n)$ codes such that $\varepsilon_n \to 0$ as $n \to \infty$. Consider the code of block size n in this sequence. Let $|\varphi\rangle = |\Phi\rangle^{R_1 A_1} \otimes |\Phi\rangle^{\tilde{A}_1 \tilde{B}_1} \otimes |\Phi\rangle^{R_1 A_1} \otimes |\Phi\rangle^{\tilde{A}_1 \tilde{B}_1}$ be the input state as in theorem 1, $W^{A_1 A_2 \tilde{A}_1 \tilde{A}_2 \to A'^{\otimes n} D}$ be the encoding isometry, and let $|\psi\rangle^{R_1 R_2 B_1^{\otimes n} B_2^{\otimes n} \tilde{B}_1 \tilde{B}_2 E^{\otimes n}} =$

$U_{\mathcal{N}}^{\otimes n} W|\varphi\rangle$. As usual, we will evaluate entropic quantities with respect to $|\psi\rangle$.

Given that Bob 1 must be able to recover a system which purifies R_1 from $B_1^{\otimes n}$ and $\tilde{B}_1$, we have by Fannes' inequality [23] that $I(R_1; B_1^{\otimes n}\tilde{B}_1) \geqslant 2\log|R_1| - n\delta_n$, where $\delta_n \to 0$ as $n \to \infty$, and likewise for Bob 2. We also have

$$\begin{aligned} I(R_1; B_1^{\otimes n}\tilde{B}_1) &= H(R_1) + H(B_1^{\otimes n}\tilde{B}_1) - H(R_1 B_1^{\otimes n}\tilde{B}_1) \\ &\leqslant H(R_1) + H(B_1^{\otimes n}) \\ &\quad + H(\tilde{B}_1) - H(R_1 B_1^{\otimes n}\tilde{B}_1) \\ &= H(R_1\tilde{B}_1) + H(B_1^{\otimes n}) - H(R_1 B_1^{\otimes n}\tilde{B}_1) \\ &= I(R_1\tilde{B}_1; B_1^{\otimes n}) \end{aligned} \tag{17}$$

where the second line follows from subadditivity, and the third line from the fact that R_1 and $\tilde{B}_1$ are in a product state. Hence, $I(R_1\tilde{B}_1; B_1^{\otimes n}) \geqslant 2\log|R_1| - n\delta_n$ and likewise, $I(R_2\tilde{B}_2; B_2^{\otimes n}) \geqslant 2\log|R_2| - n\delta_n$. Now, if we identify $R_1\tilde{B}_1$ as A_1 and $R_2\tilde{B}_2$ as A_2, we see that

$$Q_1 \leqslant \frac{1}{2n} I(A_1; B_1^{\otimes n}) + \delta_n \tag{18}$$

$$Q_2 \leqslant \frac{1}{2n} I(A_2; B_2^{\otimes n}) + \delta_n \tag{19}$$

where $\delta_n \to 0$ as $n \to \infty$. Since $I(A_1; A_2) = 0$, this rate point is clearly inside the region in equation (16), and it follows that this is indeed the capacity of the channel. ∎

An analogous theorem can easily be shown to hold for the unassisted capacity:

Theorem 4: The unassisted capacity region of a quantum broadcast channel $\mathcal{N}^{A'\to B_1 B_2}$ is the convex hull of the union of all rate points (Q_1, Q_2) satisfying

$$\begin{aligned} Q_1 &\leqslant \frac{1}{2n} I(A_1\rangle B_1^{\otimes n}) \\ Q_2 &\leqslant \frac{1}{2n} I(A_2\rangle B_2^{\otimes n}) \\ Q_1 + Q_2 &\leqslant \frac{1}{2n}[I(A_1\rangle B_1^{\otimes n}) + I(A_2\rangle B_2^{\otimes n}) - I(A_1; A_2)] \end{aligned} \tag{20}$$

for some state of the form $|\psi\rangle^{A_1 A_2 B_1^{\otimes n} B_2^{\otimes n} D E^{\otimes n}} = U_{\mathcal{N}}^{\otimes n}|\phi\rangle^{A_1 A_2 A'^{\otimes n} D}$, where $|\phi\rangle$ is a pure state.

While one might conjecture that Theorem 3 characterizes the entanglement-assisted capacity region of a broadcast channel even with the restriction $n = 1$, the analogous conjecture for the unassisted capacity is false. In fact, it isn't even true for a channel with a single receiver [24].

C. Generalization to more receivers

It is possible to generalize the protocol to more than two receivers. Without going into details, it is straightforward to show that a one-shot version of the protocol holds if there are more receivers; we simply get equations of the form of equations (7) and (8) for each receiver, and then we put them together in a way that is analogous to what we have done for two receivers.

To generalize this to the i.i.d. setting, the idea is to use a multiparty version of the FQSW protocol to decouple all the $A_1 \cdots A_n$ subsystems. Thus, instead of simply having a constraint on $Q_1 + Q_2$, we get nontrivial constraints on every possible subset of receivers. The result is the following rate region:

$$\sum_{j\in\mathcal{K}} Q_j \leqslant \frac{1}{2}\left[\sum_{j\in\mathcal{K}} I(A_j; B_j) - J(A_{\mathcal{K}})\right] \tag{21}$$

where $J(A_{\mathcal{K}}) = H(A_{j_1}) + \cdots + H(A_{j_{|\mathcal{K}|}}) - H(A_{j_1}\cdots A_{j_{|\mathcal{K}|}})$, for all $\mathcal{K} = \{j_1, \cdots, j_{|\mathcal{K}|}\} \subseteq \{1, \cdots, m\}$. The mutual informations are defined on the state $|\phi^{\mathcal{N}}\rangle^{A_1\cdots A_n B_1\cdots B_n D E} = U_{\mathcal{N}}|\phi\rangle^{A_1\cdots A_n A' D}$.

VII. Discussion

We have shown that a new protocol for entanglement-assisted communication of quantum information through quantum broadcast channels can be obtained from the FQSW protocol. Our protocol achieves the following rate region for every state $|\phi\rangle^{A_1 A_2 A' D}$:

$$\begin{aligned} Q_1 &\leqslant \frac{1}{2} I(A_1; B_1)_\psi \\ Q_2 &\leqslant \frac{1}{2} I(A_2; B_2)_\psi \\ Q_1 + Q_2 &\leqslant \frac{1}{2}\left[I(A_1; B_1)_\psi + I(A_2; B_2)_\psi - I(A_1; A_2)_\psi\right]. \end{aligned} \tag{22}$$

where $|\psi\rangle^{A_1 A_2 B_1 B_2 D E} = U_{\mathcal{N}}^{A'\to B_1 B_2 E}|\phi\rangle^{A_1 A_2 A' D}$.

One interesting thing to note is the presence of a "discarded" system D in theorem 2; this is equivalent to optimizing over all mixed states $\phi^{A_1 A_2 A'}$ rather than over pure states only. This is normally not required for most theorems in quantum information theory, but we have not found a way to prove the regularized converse without allowing for the possibility of mixed states. We thus leave it as an open problem to determine whether it is possible to demonstrate a converse theorem that does not require allowing mixed states.

Another interesting thing to note is that the rate region we obtain in the entanglement-assisted case (equation (9)) is very similar to Marton's region for classical broadcast channels (equation (1)) [3]. In fact, except for the factors of $1/2$, the two expressions are identical. This result further reinforces the analogy between entanglement-assisted quantum communication and classical information: indeed, for both the regular point-to-point quantum channel [25] and the quantum multiple-access channel [26] [27], the known achievable rate regions for entanglement-assisted quantum communication are identical to their classical counterparts.

Interestingly, the entanglement-assisted quantum capacity of point-to-point quantum channels is one of the rare quantum channel capacities that is known to be additive; furthermore, the sum of the rates possible for all senders in the entanglement-assisted capacity of quantum multiple-access channels is also known to be additive, as they are in the classical case.

All these apparent coincidences suggest a fundamental question. How far does this analogy lead: to what extent does the addition of free entanglement make quantum information theory similar to classical information theory?

Acknowledgments

The authors would like to thank Gilles Brassard, Igor Devetak, Young-Han Kim, Ivan Savov, Andreas Winter and Jon Yard for conversations that helped them in this research. They are also grateful for support from CIAR, the Canada Research Chairs program, FQRNT, MITACS and NSERC.

APPENDIX I
ASYMPTOTIC EQUALITIES

Here we formally define the asymptotic equalities involving the $\approx_{(a)}$ relation. Let $\psi = \left\{\psi_{(1)}, \psi_{(2),\ldots}\right\}$ and $\varphi = \left\{\varphi_{(1)}, \varphi_{(2),\ldots}\right\}$ be two families of quantum states, where $\psi_{(n)}$ and $\varphi_{(n)}$ are defined on a Hilbert space $\mathcal{H}^{\otimes n}$. Then we say that $\psi \approx_{(a)} \varphi$ if $\lim_{n\to\infty} \left\|\psi_{(n)} - \varphi_{(n)}\right\| = 0$. We then say that ψ and φ are asymptotically equal. Note that $\approx_{(a)}$ is transitive by the triangle inequality.

It should be mentioned that throughout the paper, asymptotic families of states are never explicitly referred to as such, but generally speaking, whenever a state depends on the number of copies, it should be considered as a family of states. In addition, with a slight abuse of notation, we allow quantum operations on families of states; it should be fairly clear which operation is done on each member of the family.

APPENDIX II
TYPICAL SUBSPACES

Much of information theory relies on the concept of typical sequences. Let $\mathcal{X}$ be some alphabet and let X be a random variable defined on $\mathcal{X}$ and distributed according to $p(x)$. Define the ε-typical set as follows:

$$\mathcal{T}_{\varepsilon}^{(n)} = \left\{ x^n \in \mathcal{X}^n \middle| \left| -\tfrac{1}{n} \log \Pr\{X^n = x^n\} - H(X) \right| \leqslant \varepsilon \right\}$$

where X^n refers to n independent, identically-distributed copies of X. It can be shown that the two following properties hold:

1) There exists a function $\varepsilon(n)$ such that $\lim_{n\to\infty} \varepsilon(n) = 0$ and such that $\Pr\{X^n \in \mathcal{T}_{\varepsilon(n)}^{(n)}\} \geqslant 1 - \varepsilon(n)$.
2) There exists an n_0 such that for all $n > n_0$, $|\mathcal{T}_{\varepsilon}^{(n)}| \leqslant 2^{n[H(X)+\varepsilon]}$.

The quantum generalization of these concepts is relatively straightforward: let $\rho^A = \sum_{x\in\mathcal{X}} p(x)|x\rangle\langle x|$ be the spectral decomposition of a quantum state ρ^A on a quantum system A. Then we can define the typical projector on the quantum system $A^{\otimes n}$ as follows:

$$\Pi_{\varepsilon}^{(n)} = \sum_{x^n \in \mathcal{T}_{\varepsilon}^{(n)}} |x^n\rangle\langle x^n|$$

We call the support of $\Pi_{\varepsilon}^{(n)}$ the typical subspace of $A^{\otimes n}$. The two properties given above generalize to the quantum case:

1) There exists a function $\varepsilon(n)$ such that $\lim_{n\to\infty} \varepsilon(n) = 0$ and such that $\operatorname{Tr}\left[\Pi_{\varepsilon(n)}^{(n)} \rho^{A^{\otimes n}}\right] \geqslant 1 - \varepsilon(n)$.
2) There exists an n_0 such that for all $n > n_0$, $\operatorname{Tr}[\Pi_{\varepsilon}^{(n)}] \leqslant 2^{n[H(A)+\varepsilon]}$.

Note that the first of these two properties implies that $\Pi_{\varepsilon(n)}^{(n)} \cdot \rho^{A^{\otimes n}} \approx_{(a)} \rho^{A^{\otimes n}}$, via the "gentle measurement" lemma (Lemma 9 in [28]). One can also easily show that the normalized version of $\Pi_{\varepsilon(n)}^{(n)} \cdot \rho^{A^{\otimes n}}$ is also asymptotically equal to $\rho^{A^{\otimes n}}$, and that it also holds for i.i.d. states with more than one subsystem.

REFERENCES

[1] T. Cover, "Broadcast channels," *IEEE Transactions on Information Theory*, vol. 18, pp. 2–14, 1972.
[2] T. Cover and J. Thomas, *Elements of Information Theory*. John-Wiley and Sons, 1991.
[3] K. Marton, "A coding theorem for the discrete memoryless broadcast channel," *IEEE Transactions on Information Theory*, vol. IT-25, pp. 306–311, 1979.
[4] J. Yard, P. Hayden, and I. Devetak, "Quantum broadcast channels," quant-ph/0603098.
[5] M. Demianowicz and P. Horodecki, "Capacity regions for multiparty quantum channels," quant-ph/0603112.
[6] ——, "Quantum channel capacities - multiparty communication," quant-ph/0603106.
[7] J. Yard, I. Devetak, and P. Hayden, "Capacity theorems for quantum multiple access channels: Classical-quantum and quantum-quantum capacity regions," quant-ph/0501045.
[8] D. Leung, J. Oppenheim, and A. Winter, "Quantum network communication – the butterfly and beyond," quant-ph/0608223.
[9] M. Hayashi, K. Iwama, H. Nishimura, R. Raymond, and S. Yamashita, "Quantum network coding," quant-ph/0601088.
[10] A. Winter, "The capacity of the quantum multiple access channel," *IEEE Trans. Info. Theory*, vol. 47, pp. 3059–3065, 2001, quant-ph/9807019.
[11] G. Klimovitch, "On the classical capacity of a quantum multiple access channel," *Proc. IEEE Intern. Sympos. Info. Theory*, p. 278, 2001.
[12] J. A. Smolin, F. Verstraete, and A. Winter, "Entanglement of assistance and multipartite state distillation," *Phys. Rev. A*, vol. 72, no. 5, pp. 052 317–+, Nov. 2005, quant-ph/0505038.
[13] I. Devetak, A. Harrow, and A. Winter, "A family of quantum protocols," *Phys. Rev. Lett.*, vol. 93, no. 230504, 2003, quant-ph/0308044.
[14] A. Abeyesinghe, I. Devetak, P. Hayden, and A. Winter, "The mother of all protocols: Restructuring quantum information's family tree," quant-ph/0606225.
[15] I. Devetak, "A triangle of dualities: reversibly decomposable quantum channels, source-channel duality, and time reversal," 2005, arXiv.org:quant-ph/0505138.
[16] C. H. Bennett, I. Devetak, A. W. Harrow, P. W. Shor, and A. Winter, "The Quantum Reverse Shannon Theorem," 2006, in preparation.
[17] S. Lloyd, "Capacity of the noisy quantum channel," *Phys. Rev. A*, no. 55:1613, 1996, quant-ph/9604015.
[18] P. Shor, "The quantum channel capacity and coherent information," *Lecture notes, MSRI workshop on quantum computation*, 2002, available online at http://www.msri.org/publications/ln/msri/2002/quantumcrypto/shor/1/.
[19] I. Devetak, "The private classical capacity and quantum capacity of a quantum channel," *IEEE Trans. Info. Theory*, no. 51(1):44, 2005, quant-ph/0304127.
[20] I. Devetak and A. Winter, "Distillation of secret key and entanglement from quantum states," *Proc. R. Soc. Lond. A*, no. 461, pp. 207–237, 2005, quant-ph/0306078.
[21] A. Uhlmann, "The 'transition probability' in the state space of a *-algebra," *Rep. Math. Phys.*, no. 9:273, 1976.
[22] I. Devetak, A. Harrow, and A. Winter, "A resource framework for quantum Shannon theory," 2005, quant-ph/0512015.
[23] M. Fannes, "A continuity property of the entropy density for spin lattices," *Commun. Math. Phys.*, vol. 31, pp. 291–294, 1973.
[24] D. P. DiVincenzo, P. W. Shor, and J. A. Smolin, "Quantum-channel capacity of very noisy channels," *Phys. Rev. A*, vol. 57, pp. 830–839, Feb. 1998, quant-ph/9706061.
[25] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal, "Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem," *IEEE Trans. Info. Theory*, vol. 48:10, pp. 2637–2655, 2002, quant-ph/0106052.
[26] M.-H. Hsieh, I. Devetak, and A. Winter, "Entanglement-assisted capacity of quantum multiple-access channels," quant-ph/0511228.
[27] M. Horodecki, J. Oppenheim, and A. Winter, "Quantum state merging and negative information," 2005, arXiv.org:quant-ph/0512247.
[28] A. Winter, "Coding theorem and strong converse for quantum channels," *IEEE Trans. Info. Theory*, vol. 45:07, pp. 2481–2485, 1999.